Correlation and spectral properties of higher-dimensional paperfolding and Rudin–Shapiro sequences

2005 J. Phys. A: Math. Gen. 38 2599

(http://iopscience.iop.org/0305-4470/38/12/005)

INSTITUTE OF PHYSICS PUBLISHING JOURNAL OF PHYSICS A: MATHEMATICAL AND GENERAL

J. Phys. A: Math. Gen. **38** (2005) 2599–2622 doi:10.1088/0305-4470/38/12/005

Correlation and spectral properties of higher-dimensional paperfolding and Rudin–Shapiro sequences

A Barbé and F von Haeseler

Department of Electrical Engineering, KU Leuven, Kasteelpark Arenberg 10, B 3001 Leuven, Belgium

E-mail: andre.barbe@esat.kuleuven.ac.be and friedrich.vonhaeseler@esat.kuleuven.ac.be

Received 6 December 2004, in final form 28 January 2005
Published 9 March 2005
Online at stacks.iop.org/JPhysA/38/2599

Abstract

We consider higher-dimensional generalizations of the classical one-dimensional 2-automatic paperfolding and Rudin–Shapiro sequences on $\mathbb{N}$. This is done by considering the same automaton-structure as in the one-dimensional case, but using binary number systems in $\mathbb{Z}^m$ instead of in $\mathbb{N}$. The correlation function and the diffraction spectrum for the resulting m-dimensional paperfolding and Rudin–Shapiro point sets are calculated through the corresponding sequences with values ± 1. They are shown to be quasi-independent of the dimension m and of the particular binary number system under consideration. It is shown that any paperfolding sequence thus obtained has a discrete spectrum. The Rudin–Shapiro sequences have an absolutely continuous Lebesgue spectral measure.

PACS numbers: 61.44.Br, 61.50.Ah, 89.75.Kd
Mathematics Subject Classification: 11B85, 62M15, 82D25

1. Introduction and preliminaries

The 2-automatic one-dimensional Thue–Morse, paperfolding and Rudin–Shapiro sequences as maps $f : \mathbb{N} \to \{+1, -1\}$, their correlation functions and spectral properties have been dealt with in, e.g., [1, 2] and were shown to be prototype examples of sequences with singular continuous, discrete and absolutely continuous diffraction spectra, respectively. Similar properties for some higher-dimensional extensions of these sequences considered as substitution sequences are mentioned in [3–5]. Note that the spectral properties of sequences also play a crucial role in the theory of quasicrystals, e.g., [6, 7].

In [8], we presented a particular two-dimensional generalization of Thue–Morse, paperfolding and Rudin–Shapiro sequences as *automatic sets*, i.e., as maps $f : \mathbb{Z}^2 \to \{0, 1\}$.

0305-4470/05/122599+24$30.00

This generalization resulted from considering the same automaton underlying the one-dimensional versions of these sequences over $\mathbb{N}$, but using binary number representations in $\mathbb{Z}^2$ instead of in $\mathbb{N}$. This paper will discuss the correlation and the spectral properties of the paperfolding and Rudin–Shapiro sequences obtained by extending this kind of generalization to all dimensions. The spectral properties of higher-dimensional Thue–Morse sequences will be dealt with in a separate paper.

We briefly recall several characterizations of m-dimensional automatic sequences, for more information consult [9, 10]. A number system (H, W) in $\mathbb{Z}^m$ is given by an *expanding* $m \times m$ integer matrix H (i.e., all eigenvalues have absolute value greater than 1) and a corresponding *complete digit set* $W = \{0, w_1, \ldots, w_{|\det(H)|-1}\} \subset \mathbb{Z}^m$, such that any $x \in \mathbb{Z}^m$ has a unique finite (H, W)-representation

$$x = H^{n-1}\nu_n + H^{n-2}\nu_{n-1} + \cdots + H\nu_2 + \nu_1,$$

where $\nu_i \in W$ and $\nu_n \neq 0$. Note that $\cup_{x\in\mathbb{Z}^m, w\in W}(Hx + w) = \mathbb{Z}^m$.

The (H, w)-decimation, or decimation for short, of a sequence $f : \mathbb{Z}^m \to \mathbb{C}$, for $w \in W$ is the sequence $\partial_{H,w}(f) : \mathbb{Z}^m \to \mathbb{C}$ defined as

$$\partial_{H,w}(f)(x) = f(Hx + w),$$

and we agree that we write ∂_w if H and W are clear from the context. Repeated application of decimations to f leads to

$$\partial_{\nu_n} \circ \partial_{\nu_{n-1}} \cdots \circ \partial_{\nu_1}(f)(x) = f(H^n x + H^{n-1}\nu_n + H^{n-2}\nu_{n-1} + \cdots + \nu_1). \tag{1}$$

The (H, W)-*kernel* of f, denoted as $\ker(f) = \ker_{H,W}(f)$ is the set of all possible decimations of f, together with f. A sequence f is (H, W) automatic if $\ker(f)$ is finite. A set $D \subset \mathbb{Z}^m$ is automatic if its characteristic sequence $\chi_D : \mathbb{Z}^m \to \{0, 1\}$, i.e., $\chi_D(x) = 1$ if and only if $x \in D$, is automatic.

Alternatively, an automatic sequence f is defined by a so-called *kernel graph*. This is a labelled directed graph where the vertices are the sequences in $\ker(f)$ and with a directed edge labelled $w \in W$ pointing from vertex g to vertex h if and only if $\partial_w(g) = h$. This graph can also be described by the *decimation matrices* $A_w \in \{0, 1\}^{\ker(f)\times\ker(f)}$, $w \in W$, i.e., matrices whose rows and columns are labelled by the kernel elements of f

$$A_w = \left(a^w_{g,h}\right),$$

where $a^w_{g,h} = 1$ if $\partial_w(g) = h$ and $a^w_{g,h} = 0$ otherwise.

For convenience, we will from here on always consider $\ker(f)$ to be a set with a fixed order that can be used for ordered labellings.

The kernel graph can also be interpreted as a finite automaton that generates the sequence f as follows: if $x \in \mathbb{Z}^m$, $x \neq 0$ has the (H, W)-representation

$$x = \sum_{j=1}^{n} H^{j-1}\nu_j,$$

then x defines a path in the kernel graph. The path begins in f, follows the arrows labelled $\nu_1, \nu_2, \ldots, \nu_n$ and terminates in an element $g \in \ker(f)$. The value of f at x is equal to the value of g at 0, i.e., $f(x) = g(0)$. This follows from (1) with $x = 0$.

Figure 1 displays the kernel graphs, corresponding decimation matrices and values for the kernel elements at 0 for the paperfolding and Rudin–Shapiro sequences with values in $\{-1, 1\}$. The underlying number system (H, W) consists of an expanding matrix H with $|\det(H)| = 2$, and a complete digit set $W = \{0, w\}$. As the complete digit set contains only two elements, (H, W) is called a *binary number system* [11].

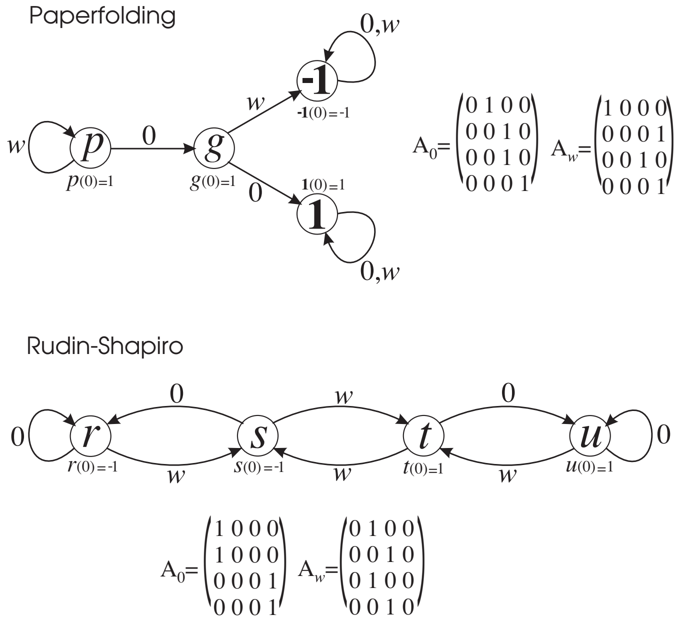


Figure 1. Kernel graphs and decimation matrices for the class of paperfolding and Rudin–Shapiro sequences.

It turns out that an automatic sequence f can also be obtained as a component of the fixed point of a substitution Σ_f on the set of vector sequences $F : \mathbb{Z}^m \to \mathbb{C}^{\ker(f)}$, defined as

$$\Sigma_f(F)(Hx + w) = A_w F(x)$$

for $w \in W, A_w$ the corresponding decimation matrix, and for all $x \in \mathbb{Z}^m$. Then, $\mathcal{F}(x) = (g(x))_{g\in\ker(f)}$ is a fixed point of this substitution, i.e., $\Sigma_f(\mathcal{F}) = \mathcal{F}$. By repeated application of the previous equation, we obtain that

$$\mathcal{F}(x) = A_{v_1} A_{v_2} \cdots A_{v_n} \mathcal{F}(0), \tag{2}$$

where $x = \sum_{j=1}^{n} H^{j-1} v_j$ is the unique (H, W)-representation of $x \in \mathbb{Z}^m \backslash \{0\}$. Note that $\mathcal{F}(0) = A_0 \mathcal{F}(0)$, and that every fixed point $G(0)$ of A_0 gives, via equation (2), rise to a sequence $G : \mathbb{Z}^m \to \mathbb{C}^{\ker(f)}$ such that the component of G with label f has the kernel graph of f as a generating automaton. However, it is not necessarily true that the kernel graph of this component is the same as the kernel graph of f. For example, in figure 1, the values at 0 of the kernel elements correspond to one such fixed point $\mathcal{F}_0$ when the sequences take values in $\{-1, 1\}$. By letting all values at 0 be equal to 1, what also corresponds to a fixed point of A_0, the automaton would generate the constant sequence **1**. But then the figures do no longer represent the kernel graph of this sequence **1**.

Observe that for the kernel graphs in figure 1, no reference is made to a specific dimension m, and neither to a specific underlying binary number system $(H, W = \{0, w\})$ in $\mathbb{Z}^m$. Any appropriate choice of these parameters corresponds to a particular paperfolding or Rudin–Shapiro sequence. Hence, these kernel graphs represent a whole class of *automaton-similar* sequences. We formalize this concept in

Definition 1.1. *Let* $f_1 : \mathbb{Z}^{m_1} \to \mathbb{C}$ *be* (H_1, W_1)*-automatic and let* $f_2 : \mathbb{Z}^{m_2} \to \mathbb{C}$ *be* (H_2, W_2)*-automatic. The sequences* f_1 *and* f_2 *are called automaton-similar if there exist bijections*

$\vartheta : \ker_{H_1,W_1}(f_1) \to \ker_{H_2,W_2}(f_2)$ and $\xi : W_1 \to W_2$ such that

(1) $\vartheta(f_1) = f_2$,
(2) $\vartheta(\partial_{H_1,w}(g)) = \partial_{H_2,\xi(w)}(\vartheta(g))$ for all $w \in W_1$, $g \in \ker(f_1)$,
(3) $\xi(0) = 0$.

Example. Let $f : \mathbb{Z}^m \to \mathbb{C}$ be a (H, W)-automatic sequence and let $\theta : \mathbb{C} \to \mathbb{C}$ be an injective map. The sequence $\theta(f)$ defined as $\theta(f)(x) = \theta(f(x))$ is automaton-similar to f. In particular, if θ is the multiplication with a nonzero number α, then the sequence $(\alpha f(x))_{x\in\mathbb{Z}^m}$ is automaton-similar to f. Furthermore, note that if g is (H, W)-automatic and automaton-similar to a paperfolding sequence f with values ± 1, then g has two different values a, b. Moreover, replacing a by -1 and b by 1 generates a (H, W)-automatic sequence g' which is paperfolding and has values ± 1. This also shows that every sequence g in the automaton-similarity class of paperfolding sequences can be obtained from a paperfolding sequence with values ± 1. The same holds for sequences which are automaton-similar to a Rudin–Shapiro sequence with values ± 1.

In other words, automaton-similar sequences have isomorphic kernel graphs. Practically, it is convenient to consider $\vartheta : \ker_{H_1,W_1}(f_1) \to \ker_{H_2,W_2}(f_2)$ as an order preserving map. Then $A_{\xi(w)}$ for f_2 is the same as A_w for f_1.

In that way, the class of automaton-similar paperfolding and Rudin–Shapiro sequences is given by all sequences generated by the automaton corresponding to the kernel graphs in figure 1, for all dimensions m and all binary number systems (H, W) in $\mathbb{Z}^m$.

A complete characterization of binary number systems for $\mathbb{Z}^m$ is given in [11]. We recall the notion of the companion matrix C_H of a matrix $H \in \mathbb{Z}^{m\times m}$. When $\chi_H(x) = x^m + a_{m-1}x^{m-1} + \cdots + a_0$ denotes the characteristic polynomial of H, then the companion matrix is given by

$$C_H = \begin{pmatrix} 0 & 0 & 0 & \cdots & 0 & -a_0 \\ 1 & 0 & 0 & \cdots & 0 & -a_1 \\ 0 & 1 & 0 & \cdots & 0 & -a_2 \\ 0 & 0 & 1 & \cdots & 0 & -a_3 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & 1 & -a_{m-1} \end{pmatrix}.$$

For binary number systems, needing an expanding matrix $H \in \mathbb{Z}^{m\times m}$ with $|\det(H)| = 2$, it was shown (theorem 3.4 in [11]) that H has a complete digit set *if and only if*

(a) H is $\mathbb{Z}$-similar to its companion matrix C_H, i.e., there exists a matrix $Q \in \mathrm{GL}(m, \mathbb{Z})$, the general linear group of $m \times m$ integer matrices with determinant ± 1, such that $H = QC_HQ^{-1}$.
and
(b) $\{0, e_1\}$ is a complete digit set for C_H, where $e_1 = (\underbrace{1, 0, 0, \ldots, 0}_{m})^T$, ^{T} denoting transposition.

Moreover, it was shown that for any such C_H for which $\{0, e_1\}$ is a complete digit set, there exists a finitely generated commutative subgroup $\mathsf{G}(C_H) \subset \mathrm{GL}(m, \mathbb{Z})$ such that every complete digit set for C_H is given as $\{0, Ge_1\}$ with $G \in \mathsf{G}(C_H)$. Moreover, if $G \in \mathsf{G}(C_H)$, then

$$|\det(G)| = 1 \qquad \text{and} \qquad GC_H = C_HG.$$

In turn, all complete digit sets for $H = QC_HQ^{-1}$ are given by $W = \{0, QGe_1\}$, $G \in \mathsf{G}(C_H)$. Thus, it is clear that all

$$[H, W] = \{(QC_HQ^{-1}, \{0, QGe_1\}) | Q \in \mathrm{GL}(m, \mathbb{Z})\}, G \in \mathsf{G}(C_H)\}$$

form a class of $\mathbb{Z}$-*similar* binary number systems. We call $(C_H, \{0, e_1\})$ the *companion representative* of this class. If (H_1, W_1) and (H_2, W_2) are in the same class, we denote this by

$$(H_1, W_1) \overset{\mathbb{Z}}{\sim} (H_2, W_2).$$

Corollary 1.2. *Let* $(H_i, \{0, w_i\})$, $i = 1, 2$ *be binary number systems for* $\mathbb{Z}^m$. $(H_1, \{0, w_1\}) \overset{\mathbb{Z}}{\sim} (H_2, \{0, w_2\})$ *if and only if there exist* $Q, Q_1 \in \mathrm{GL}(m, \mathbb{Z})$ *such that*

$$(H_2, \{0, w_2\}) = (QH_1Q^{-1}, \{0, QUw_1\})$$

with $U = Q_1GQ_1^{-1}$ *and* $G \in \mathsf{G}(C_H)$. *Moreover, it holds that* $H_1U = UH_1$.

Proof.

(i) if: $(H_1, \{0, w_1\})$ and $(H_2, \{0, w_2\})$ belong to the same class then there exist a matrix C_H and matrices $Q_i \in \mathrm{GL}(m, \mathbb{Z})$ such that $H_i = Q_iC_HQ_i^{-1}$, $i = 1, 2$. Let $Q = Q_2Q_1^{-1}$, then one computes that $H_2 = QH_1Q^{-1}$.
Furthermore, one has $w_1 = Q_1G_1e_1$ and $w_2 = Q_2G_2e_1$, where $G_i \in \mathsf{G}(C_H)$, $i = 1, 2$. This yields $w_2 = Q_2G_2G_1^{-1}Q_1^{-1}w_1$, and using the fact that $Q = Q_2Q_1^{-1}$, one gets $w_2 = Q_2Q^{-1}Q_1G_2G_1^{-1}Q_1^{-1} = QUw_1$, as desired.

(ii) only if: since $(H_1, \{0, w_1\})$ is a binary number system there exists $Q_1 \in \mathrm{GL}(m, \mathbb{Z})$ and a companion matrix C_H such that $H_1 = Q_1C_HQ_1^{-1}$ and $w_1 = Q_1Ge_1$, with $G \in \mathsf{G}(C_H)$. Let $H_2 = QH_1Q^{-1}$ and $w_2 = QUw_1$, with $Q, Q_1 \in \mathrm{GL}(m, \mathbb{Z})$ and $U \in Q_1\mathsf{G}(C_H)Q_1^{-1}$. Then certainly, $H_2 = (QQ_1)C_H(QQ_1)^{-1}$ and using the expression for w_1, we get

$$w_2 = QUw_1 = QQ_1GQ_1^{-1}w_1 = QQ_1GQ_1^{-1}(Q_1G_1e_1) = (QQ_1)(GG_1)e_1,$$

with $(GG_1) \in \mathsf{G}(C_H)$ and $(QQ_1) \in \mathrm{GL}(m, \mathbb{Z})$. This proves that $(H_2, \{0, w_2\})$ belongs to the same class as $(H_1, \{0, w_1\})$.

(iii) That $H_1U = UH_1$ follows from the fact that $C_HG = GC_H$ or, using $U = Q_1GQ_1^{-1}$ which is equivalent to $G = Q_1^{-1}UQ_1$, that $\big(Q_1C_HQ_1^{-1}\big)U = U\big(QC_HQ_1^{-1}\big)$. Then use the fact that $Q_1C_HQ_1^{-1} = H_1$. □

In [11] one finds a list of binary number systems $(C_H, \{0, e_1\})$ and the corresponding generators for the group G for dimensions $m = 1, 2, \ldots, 6$. Let us just mention that the number of $\mathbb{Z}$-similar binary number system classes equals 1, 4, 4, 12, 7, 25 for dimension $m = 1, 2, \ldots, 6$, respectively. For $m = 1, 2$, every $\mathbb{Z}$-similar class has only finitely many complete digit sets, since $\mathsf{G}(C_H)$ is finite. If $m \geqslant 3$, then $\mathsf{G}(C_H)$ is infinite and therefore one has infinitely many complete digit sets for every class.

The four $\mathbb{Z}$-similarity classes of binary number systems for $\mathbb{Z}^2$, given by their companion representatives, are

$$\begin{aligned} C_1 &= \begin{pmatrix} 0 & -2 \\ 1 & -2 \end{pmatrix} \qquad C_2 = \begin{pmatrix} 0 & -2 \\ 1 & -1 \end{pmatrix} \\ C_3 &= \begin{pmatrix} 0 & -2 \\ 1 & 0 \end{pmatrix} \qquad C_4 = \begin{pmatrix} 0 & -2 \\ 1 & 1 \end{pmatrix} \end{aligned} \tag{3}$$

and with complete digit set $E = \{0, e_1\} = \{(0, 0)^T, (1, 0)^T\}$.

We will also consider the $\mathbb{Z}^2$-binary number system

$$(H, E) = \left(\begin{pmatrix} -1 & -1 \\ 1 & -1 \end{pmatrix}, E \right), \tag{4}$$

which is $\mathbb{Z}$-similar to (C_1, E) as follows:

$$(H, E) = \left(Q_H C_1 Q_H^{-1}, \{0, Q_H e_1\} \right), \qquad \text{with} \quad Q_H = \begin{pmatrix} 1 & -1 \\ 0 & 1 \end{pmatrix}. \tag{5}$$

Using the notion of $\mathbb{Z}$-similarity, we introduce a stronger notion of similarity between automatic sequences defined by binary number systems.

Definition 1.3. *Let $f_1, f_2 : \mathbb{Z}^m \to \mathbb{C}$ be (H_1, W_1)- and (H_2, W_2)-automatic sequences, respectively, with (H_1, W_1) and (H_2, W_2) binary number systems for $\mathbb{Z}^m$. f_1 and f_2 are called similar if*

(i) f_1 and f_2 are automaton-similar (see definition 1.1),
(ii) $\vartheta(g)(0) = g(0)$ for all $g \in \ker(f_1)$,
(iii) $(H_1, W_1) \overset{\mathbb{Z}}{\sim} (H_2, W_2)$.

Note that by (ii), similar sequences always have the same values. By (iii), see the later theorem 2.1, it follows that similar sequences in $\mathbb{Z}^m$ are related through a $\mathbb{Z}$-linear transformation of coordinates.

Since we are interested in the correlation and the spectral properties of the generalized paperfolding and Rudin–Shapiro sequences, we introduce some necessary notions.

For a sequence $f : \mathbb{Z}^m \to \mathbb{C}$, the (auto)correlation function γ_{ff} is generally defined as

$$\gamma_{ff}(k) = \lim_{R\to\infty} \frac{1}{\mathrm{vol}(B_R(0))} \sum_{x \in B_R(0)\cap\mathbb{Z}^m} f(x)\overline{f}(x+k), \tag{6}$$

for all $k \in \mathbb{Z}^m$, provided the limit exists. $\overline{f}$ denotes the complex conjugate of f; $\mathrm{vol}(B_R(0))$ denotes the volume of a ball of radius R centred at 0 (in a proper norm). For two sequences $g, h : \mathbb{Z}^m \to \mathbb{C}$, the correlation function is defined as

$$\gamma_{gh}(k) = \lim_{R\to\infty} \frac{1}{\mathrm{vol}(B_R(0))} \sum_{x \in B_R(0)\cap\mathbb{Z}^m} g(x)\overline{h}(x+k). \tag{7}$$

The existence of the correlation functions γ_{gh}, where g, h belong to one of the kernels of the m-dimensional paperfolding or Rudin–Shapiro sequence, has been established in [12] under the additional condition that there exists an invertible $m \times m$ matrix $P \in \mathbb{R}^{m\times m}$ such that $P^{-1}HP$ equals a block-diagonal matrix

$$\Lambda = \mathrm{diag}(\lambda_1, \lambda_2, \ldots, \lambda_s, \Lambda_1, \Lambda_2, \ldots, \Lambda_t), \tag{8}$$

where the $|\lambda_j| > 1$ correspond to the real eigenvalues of H and the Λ_j are 2×2 matrices of the form

$$\Lambda_j = \begin{pmatrix} a_j & -b_j \\ b_j & a_j \end{pmatrix},$$

where $a_j, b_j \in \mathbb{R}$ and $|\det(\Lambda_j)| = a_j^2 + b_j^2 > 1$. Λ_j corresponds to a pair of complex eigenvalues $(a_j \pm b_j i)$ of H.

Let $\underline{R} = (R_1, R_2, \ldots, R_s, R_{s+1}, \ldots, R_{s+t})$ have positive real entries, and let $\mathcal{C}(\underline{R})$ denote the cylinder

$$\mathcal{C}(\underline{R}) = \{(x_1, \ldots, x_s) \mid |x_i| \leqslant R_i, i = 1, \ldots, s\} \times \left\{(x_1, y_1, \ldots, x_t, y_t) \middle| x_j^2 + y_j^2 \leqslant R_{s+j}^2, j = 1, \ldots, t\right\}.$$

$P\mathcal{C}(\underline{R})$ is the cylinder resulting from the transformation of the cylinder $\mathcal{C}(\underline{R})$ under P. As explained in [12], it is advantageous to consider equations (6), (7) with the cylinders $P\mathcal{C}(\underline{R})$ as the proper balls, yielding

$$\gamma_{gh}(k) = \lim_{\underline{R}\Rightarrow\infty} \frac{1}{\mathrm{vol}(P\mathcal{C}(\underline{R}))} \sum_{x\in B_R(0)\cap\mathbb{Z}^m} g(x)\overline{h}(x+k),$$

where $\underline{R} \Rightarrow \infty$ means that all R_i go simultaneously but independently to ∞.

From now on we assume that every expanding H with $|\det(H)| = 2$ is equivalent to a diagonal matrix as in equation (8).

Remark. For all binary number systems $(C_H, \{0, e_1\})$ listed in [11] there exists a matrix $P \in \mathbb{R}^{m\times m}$ such that PC_HP^{-1} is of the required diagonal form (8). This is a consequence of the fact that the characteristic polynomials of these matrices C_H all have simple roots (in $\mathbb{C}$). This is also the case for $(H, W) \overset{\mathbb{Z}}{\sim} (C_H, \{0, e_1\})$ as H has the same characteristic equation as C_H.

We now recall some elementary facts concerning the spectrum of an m-dimensional sequence.

We consider the correlation function, provided it exists, as a weighted Dirac comb (or tempered distribution) in $\mathbb{R}^m$, see [13] chapter 6, i.e.,

$$\gamma_{ff} := \sum_{x\in\mathbb{Z}^m} \gamma_{ff}(x)\delta_x,$$

where δ_x is the Dirac impulse at x. Viewing the correlation function as a tempered distribution also justifies the name *correlation measure* for the tempered distribution γ_{ff}. Then the *spectrum* of f is the m-dimensional Fourier-transform $\hat{\gamma}_{ff}$ of the correlation measure γ_{ff}, [7]. It is itself a tempered distribution [13]. With $\nu \in \mathbb{R}^m$, this gives

$$\hat{\gamma}_{ff}(\nu) = \sum_{k\in\mathbb{Z}^m} \exp(-2\pi\mathrm{i}\,\nu^T k\gamma_{ff}(k))$$

in distribution sense. Note that $\hat{\gamma}_{ff}(\nu) = \hat{\gamma}_{ff}(\nu+\theta)$, for any $\theta \in \mathbb{Z}^m$. Thus, it suffices to consider the spectrum only for $\nu \in [0, 1)^m$, see also [14].

2. Basic relations between similar sequences and their correlations and spectra

In this section, it will be shown that there is a simple relationship between the correlation functions and spectra of similar automatic sequences.

The following theorem states that two similar sequences are $\mathbb{Z}$-linear 'rearrangements' of each other.

Theorem 2.1. *Let* $f_1, f_2 : \mathbb{Z}^m \to \mathbb{C}$ *be similar sequences that are* $(H_1, \{0, w_1\})$*- and* $(H_2, \{0, w_2\})$*-automatic, respectively. Let, according to corollary 1.2,* $(H_2, \{0, w_2\}) = (QH_1Q^{-1}, \{0, QUw_1\})$ *and let* $R = QU$*. Then,*

$$f_2(x) = f_1(R^{-1}x)\, \textit{for all}\, x \in \mathbb{Z}^m.$$

Proof. This follows from the fact that the bijections $\vartheta : \ker_{H_1,W_1}(f_1) \to \ker_{H_2,W_2}(f_2)$ and $\xi : W_1 \to W_2$ defining the automaton-similarity between f_1 and f_2 are given by

$$\vartheta : \mathbb{C}^{\mathbb{Z}^m} \to \mathbb{C}^{\mathbb{Z}^m} : \vartheta(h)(x) = h(R^{-1}x)$$

for all $h \in \ker(f_1)$, and

$$\xi : \{0, w_1\} \to \{0, w_2\} : \qquad \xi(0) = 0 \text{ and } \xi(w_1) = w_2 = Rw_1.$$

Indeed, it holds that, for all $h \in \ker(f_1)$

$$\begin{aligned}\partial_{H_2,0}(\vartheta(h))(x) &= \vartheta(h)(H_2x) = h(R^{-1}H_2x) = h(U^{-1}Q^{-1}QH_1Q^{-1}x) = h(U^{-1}H_1Q^{-1}x)\\ &= h(H_1U^{-1}Q^{-1}x) = h(H_1R^{-1}x) = \vartheta\big(\partial_{H_1,0}(h)\big)(x),\end{aligned}$$

where use has been made of the fact that $UH_1 = H_1U$ (corollary 1.2). In the same way,

$$\begin{aligned}\partial_{H_2,w_2}(\vartheta(h))(x) &= \vartheta(h)(H_2x + w_2) = h(R^{-1}H_2x + R^{-1}w_2) = h(U^{-1}Q^{-1}QH_1Q^{-1}x + w_1)\\ &= h(U^{-1}H_1Q^{-1}x + w_1) = h(H_1U^{-1}Q^{-1}x + w_1) = h(H_1R^{-1}x + w_1)\\ &= \vartheta(\partial_{H_1,w_1}(h))(x).\end{aligned}$$

□

The next theorem gives an analogous result for the correlation functions of similar sequences.

Theorem 2.2. *Let $f_1, f_2 : \mathbb{Z}^m \to \mathbb{C}$ be similar sequences that are $(H_1, \{0, w_1\})$- and $(H_2, \{0, w_2\})$-automatic, respectively. Let, according to theorem 2.1, $\vartheta : \ker_{H_1,W_1}(f_1) \to \ker_{H_2,W_2}(f_2)$ be given by*

$$\vartheta(h)(x) = h(R^{-1}x).$$

Then, it holds that

$$\gamma_{\vartheta(g)\vartheta(h)}(Rk) = \gamma_{gh}(k) \textit{ for all } k \in \mathbb{Z}^m.$$

Proof. Let $P \in \mathrm{GL}(m, \mathbb{R})$ be such that $P^{-1}HP = \Lambda$ is of the diagonal form as in equation (8). Then, recalling the definition from the introduction, the correlation function of $g, h \in \ker_{H_1,W_1}(f_1)$ is given by

$$\begin{aligned}\gamma_{gh}(k) &= \lim_{\underline{R}\Rightarrow\infty} \frac{1}{\mathrm{vol}(P\mathcal{C}(\underline{R}))} \sum_{x\in P\mathcal{C}(\underline{R})\cap\mathbb{Z}^m} g(x)\overline{h}(x+k)\\ &= \lim_{\underline{R}\Rightarrow\infty} \frac{1}{\mathrm{vol}(P\mathcal{C}(\underline{R}))} \sum_{x\in P\mathcal{C}(\underline{R})\cap\mathbb{Z}^m} \vartheta(g)(Rx)\overline{\vartheta(h)}(R(x+k))\\ &= \lim_{\underline{R}\Rightarrow\infty} \frac{1}{\mathrm{vol}(P\mathcal{C}(\underline{R}))} \sum_{y\in R(P\mathcal{C}(\underline{R})\cap\mathbb{Z}^m)} \vartheta(g)(y)\overline{\vartheta(h)}(y+Rk).\end{aligned}$$

Since $|\det(R)| = 1$, the last expression can be written as

$$\lim_{\underline{R}\Rightarrow\infty} \frac{1}{\mathrm{vol}(RP\mathcal{C}(\underline{R}))} \sum_{y\in RP\mathcal{C}(\underline{R})\cap\mathbb{Z}^m} \vartheta(h_1)(y)\overline{\vartheta(h_2)}(y+Rk). \tag{9}$$

By the fact that $R = QU$, and $UH_1 = H_1U$ (see corollary 1.2), it follows that $P^{-1}R^{-1}H_2RP = P^{-1}U^{-1}Q^{-1}H_2QUP = P^{-1}U^{-1}H_1UP = P^{-1}H_1P = \Lambda$. This shows that $(RP)^{-1}H_2RP = \Lambda$, i.e., that $RP\mathcal{C}(\underline{R})$ is a proper cylinder for H_2. Thus, expression (9) is equal to $\gamma_{\vartheta(g)\vartheta(h)}(Rk)$, and the assertion is proven. □

As far as the Fourier transform $\widehat{\gamma}_{ff}$ of the correlation measure γ_{ff} is concerned, we have

Theorem 2.3. *Let f_1 and f_2 be similar sequences that are $(H_1, \{0, w_1\})$- and $(H_2, \{0, w_2\})$-automatic, respectively, and let R be as in theorem 2.1. Then,*

$$\widehat{\gamma}_{f_2f_2}(\nu) = \widehat{\gamma}_{f_1f_1}(R^T\nu), \qquad \nu \in \mathbb{R}^m.$$

Proof. The correlation measures of f_1 and f_2 are

$$\gamma_{f_i f_i}(x) = \sum_{x \in \mathbb{Z}^m} \gamma_{f_i f_i}(x)\delta_x, \qquad i = 1, 2.$$

Due to theorem 2.2, one has $\gamma_{f_1 f_1}(k) = \gamma_{f_2 f_2}(Rk), k \in \mathbb{Z}^m$ for the correlation functions. This relation remains true if one considers the associated correlation measures $\gamma_{f_1 f_1}(x) = \gamma_{f_2 f_2}(Rx), x \in \mathbb{R}^m$. Interpreting $\gamma_{f_2 f_2}(Rx)$ as $\left(\gamma_{f_2 f_2} \circ R\right)(x)$ allows to apply formula (7), p 144 in [15]. Reinterpreting this formula in our notation, taking into account that $|\det(R)| = 1$, yields the desired assertion for the Fourier transforms. □

As a consequence of theorems 2.1, 2.2, 2.3 it is clear that it is sufficient to calculate the correlation function and spectrum of one representative in each class of similar sequences, the companion representatives for example. The correlation function and the spectrum of any similar sequence can be obtained by a linear coordinate transformation.

The next result, the proof of which is rather trivial, shows how the correlation function of a sequence changes if the values are changed by the map $\theta(z) = \alpha z + \beta, \alpha, \beta \in \mathbb{C}$, and $\alpha \neq 0$. It features the average μ_f of a sequence $f : \mathbb{Z}^m \to \mathbb{C}$, defined as the following limit (provided it exists):

$$\mu_f = \lim_{\underline{R} \Rightarrow \infty} \frac{1}{\operatorname{vol}(P\mathcal{C}(\underline{R}))} \sum_{x \in P\mathcal{C}(\underline{R}) \cap \mathbb{Z}^m} f(x).$$

Lemma 2.4. *Let* $f : \mathbb{Z}^m \to \mathbb{C}$, *and let* $g(x) = \alpha f(x) + \beta, x \in \mathbb{Z}^m$. *Then,*

$$\gamma_{gg}(k) = \alpha^2 \gamma_{ff}(k) + (2\alpha\beta\mu_f + \beta^2) \tag{10}$$

$$\widehat{\gamma}_{gg} = \alpha^2 \widehat{\gamma}_{ff} + (2\alpha\beta\mu_f + \beta^2) \sum_{\nu \in \mathbb{Z}^m} \delta_\nu. \tag{11}$$

Note that g is automaton-similar to f. In particular, if $g = -f$, then the above lemma shows that g has the same correlation function as f.

We recall from [12] how to compute the correlation function of an (H, W)-automatic sequence f when (H, W) is a binary number system. If $\gamma_{gh}(k)$ denotes the correlation function between any two sequences g and h in the kernel of f and $\Gamma_f(k)$ denotes the $(\ker(f) \times \ker(f))$-vector with components $\gamma_{gh}, g, h \in \ker(f)$, the correlation function satisfies the equations

$$\Gamma_f(Hk) = \tfrac{1}{2}(A_0 \otimes A_0 + A_w \otimes A_w)\Gamma_f(k), \tag{12}$$

$$\Gamma_f(Hk + w) = \tfrac{1}{2}(A_0 \otimes A_w)\Gamma_f(k) + \tfrac{1}{2}(A_w \otimes A_0)\Gamma_f(k + 2H^{-1}w), \tag{13}$$

for all $k \in \mathbb{Z}^m$. In this equation, A_0 and A_w are the decimation matrices, and $\otimes$ denotes the Kronecker product. In [12], it was also shown that $\Gamma_f(k)$ exists for all $k \in \mathbb{Z}^m$, if $\Gamma_f(0)$ exists. Also a procedure was presented to calculate $\Gamma_f(k)$ from $\Gamma_f(0)$.

3. Correlation and spectral properties of paperfolding sequences

In this section, we consider the automaton-similarity class of all ± 1-valued paperfolding sequences. That is, all sequences f which are obtained by the automaton in figure 1 together with the sequences $-f$.

Figure 2 displays the two-dimensional paperfolding sequences based on the companion representatives (C_j, E), $j = 1, 2, 3, 4$, of the four different $\mathbb{Z}$-similarity classes of binary number systems in $\mathbb{Z}^2$, see (3). We will derive an explicit formula for the correlation function

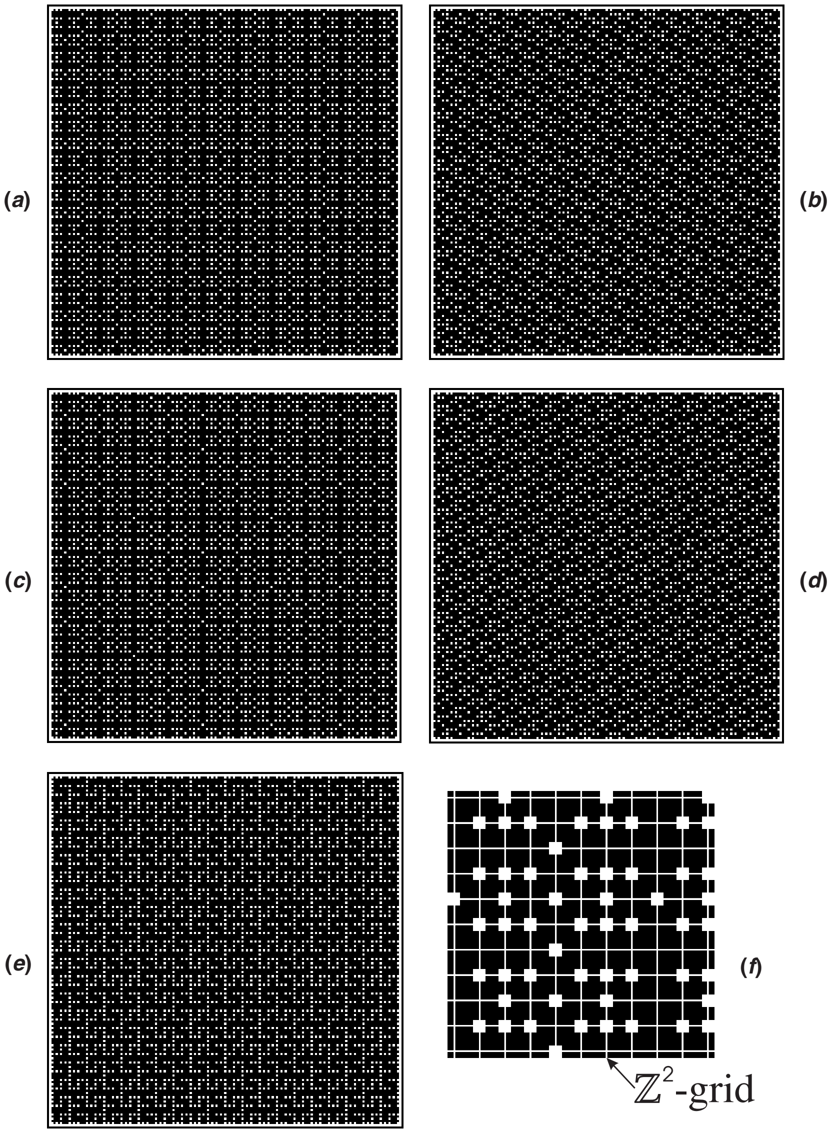


Figure 2. (a)–(d) The two-dimensional ± 1-paperfolding sequences based on the four companion $\mathbb{Z}^2$-binary number systems (C_j, E), $j = 1, 2, 3, 4$, in that order. The display domain is $[-40, 40]^2$. White pixels on the $\mathbb{Z}^2$-grid, see the enlargement (f), correspond to +1, grid points without a white pixel correspond to -1 (or 0 in case $\{0, 1\}$-paperfolding sets are considered). Part(e) displays the paperfolding sequence for the number system $(H, E) \overset{\mathbb{Z}}{\sim} (C_1, E)$ (see (4)). By application of theorem 2.1, it can be obtained from (a) by the $\mathbb{Z}$-linear coordinate transformation $R = Q_H$ in equation (5). Observe that, although (a) and (c) look very similar, they cannot be transformed in each other. Neither can (d) and (b), although (d) looks similar to a left–right reflection of (b).

of all ± 1-valued paperfolding sequences. Furthermore, we will show that the spectrum, i.e., the Fourier transform of the correlation measure, is purely discrete and of the form

$$\sum_{n=2}^{\infty} c_n \sum_{\nu \in \mathcal{L}_n} \delta_\nu,$$

where c_n depends only on n, i.e., not on H or W or the dimension m. The sets $\mathcal{L}_n$ are m-dimensional point lattices depending on H alone.

The kernel elements for the ± 1 paperfolding sequence p are given by $\{p, g, \mathbf{1}, -\mathbf{1}\}$, see figure 1, where $\pm\mathbf{1}$ are the constant sequences with value 1 and -1, respectively. The corresponding Γ_p in equations (12), (13) is the 16-component vector

$$(\gamma_{pp}, \gamma_{pg}, \gamma_{p\mathbf{1}}, \gamma_{p(-\mathbf{1})}, \gamma_{gp}, \gamma_{gg}, \gamma_{g\mathbf{1}}, \gamma_{g(-\mathbf{1})}, \ldots, \gamma_{(-\mathbf{1})p}, \gamma_{(-\mathbf{1})g}, \gamma_{(-\mathbf{1})\mathbf{1}}, \gamma_{(-\mathbf{1})(-\mathbf{1})})^T.$$

Since the constant sequences $\mathbf{1}$ and $-\mathbf{1}$ form a sink of the kernel graph, theorem 3.6 in [12] implies that all these correlation functions exist. Indeed, we obviously have, for all $k \in \mathbb{Z}^m$,

$$\gamma_{\mathbf{11}}(k) = \gamma_{(-\mathbf{1})(-\mathbf{1})}(k) = 1 \qquad \text{and} \qquad \gamma_{\mathbf{1}(-\mathbf{1})}(k) = \gamma_{(-\mathbf{1})\mathbf{1}}(k) = -1.$$

Note also that, for all $k \in \mathbb{Z}^m$, $\gamma_{\mathbf{1}p}(k) = \gamma_{p\mathbf{1}}(k) = \mu_p$, the average of the sequence p, and that $\gamma_{\mathbf{1}g}(k) = \gamma_{g\mathbf{1}}(k) = \mu_g$, the average of the sequence g. Moreover, one has that $\gamma_{(-\mathbf{1})p}(k) = \gamma_{p(-\mathbf{1})}(k) = \mu_{-p} = -\mu_p$ and $\gamma_{\mathbf{1}g}(k) = \gamma_{g\mathbf{1}}(k) = \mu_{-g} = -\mu_g$.

As the paperfolding sequence takes the values ± 1, it is clear that

$$\gamma_{pp}(0) = \lim_{\underline{R}\to\infty} \frac{1}{\mathrm{vol}(P\mathcal{C}(\underline{R}))} \sum_{x \in P\mathcal{C}(\underline{R}) \cap \mathbb{Z}^m} p^2(x) = 1 \tag{14}$$

$$\gamma_{gg}(0) = \lim_{\underline{R}\to\infty} \frac{1}{\mathrm{vol}(P\mathcal{C}(\underline{R}))} \sum_{x \in P\mathcal{C}(\underline{R}) \cap \mathbb{Z}^m} g^2(x) = 1. \tag{15}$$

Simplifying equations (12) and (13) by taking into account the particular γ-values from above, these equations become

$$\gamma_{pp}(Hk) = \tfrac{1}{2}(\gamma_{pp}(k) + \gamma_{gg}(k)) \tag{16}$$

$$\gamma_{pp}(Hk + w) = 0 \tag{17}$$

$$\gamma_{gg}(Hk) = 1 \tag{18}$$

$$\gamma_{gg}(Hk + w) = -1 \tag{19}$$

$$\gamma_{gp}(Hk) = 0 \tag{20}$$

$$\gamma_{pg}(Hk) = 0 \tag{21}$$

$$\mu_p = 0 \tag{22}$$

$$\mu_g = 0. \tag{23}$$

The first four of these recursive equations allow the computation of the correlation function $\gamma_{pp}(k)$. An inspection of (17), and of (16) together with (17, 19), already indicates that γ_{pp} is constant on point lattices of the form $H^n(Hk+w)$, $k \in \mathbb{Z}^m$, $n \in \mathbb{N}$. This observation motivates the following partitioning of the point lattice $\mathbb{Z}^m$.

First observe that, because the point lattice $\mathbb{Z}^m$ partitions as

$$\mathbb{Z}^m = H\mathbb{Z}^m \cup (H\mathbb{Z}^m + w), \tag{24}$$

the point lattice $H^n\mathbb{Z}^m$ partitions as

$$H^n\mathbb{Z}^m = H^{n+1}\mathbb{Z}^m \cup (H^{n+1}\mathbb{Z}^m + H^n w). \tag{25}$$

For $n \geqslant 1$, let

$$\mathsf{L}_n = H^n\mathbb{Z}^m + H^{n-1}w.$$

Then, applying (25) iteratively, starting from (24), it follows that $\mathbb{Z}^m$ partitions as

$$\mathbb{Z}^m = \{0\} \cup \bigcup_{i=1}^{\infty} \mathsf{L}_i.$$

Theorem 3.1. *The correlation function of an (H, W)-automatic paperfolding sequence p with values ± 1 is given by*

$$\gamma_{pp}(0) = 1 \tag{26}$$

$$\gamma_{pp}(k) = 0 \qquad \text{if} \quad k \in \mathsf{L}_1 \tag{27}$$

$$\gamma_{pp}(k) = \frac{2^{n-1} - 3}{2^{n-1}} \qquad \text{if} \quad k \in \mathsf{L}_n, n \geqslant 2. \tag{28}$$

Proof. Equation (26) follows from (14) and (27) from (17).

Now, let $k \in \mathsf{L}_n$, i.e., $k = H^n l + H^{n-1} w$ for some $l \in \mathbb{Z}^m, n \geqslant 2$. Then, according to (16), we get (step 1)

$$\gamma_{pp}(k) = \tfrac{1}{2}[\gamma_{pp}(H^{n-1} l + H^{n-2} w) + \gamma_{gg}(H^{n-1} l + H^{n-2} w)].$$

The right most term equals 1, see (18). Applying (16) to the leftmost term gives (step 2)

$$\gamma_{pp}(k) = \tfrac{1}{2}\big[\tfrac{1}{2}[\gamma_{pp}(H^{n-2} l + H^{n-3} w) + \gamma_{gg}(H^{n-2} l + H^{n-3} w)] + 1\big].$$

Again, $\gamma_{gg}(H^{n-2} l + H^{n-3} w) = 1$. Iterating further, we find after step $(n-1)$

$$\gamma_{pp}(k) = \tfrac{1}{2}\big[\tfrac{1}{2}\big[\tfrac{1}{2} \cdots \tfrac{1}{2}\big[\tfrac{1}{2}[\gamma_{pp}(Hl + w) + \gamma_{gg}(Hl + w)] + 1] \cdots + 1\big] + 1\big].$$

Using the fact that $\gamma_{pp}(Hl + w) = 0$, and $\gamma_{gg}(Hl + w) = -1$, see (17) and (19), we see that this amounts to iterating the recursion $s_{t+1} = \frac{1}{2}[s_t + 1]$ for $(n-1)$ times, starting from $s_1 = \frac{1}{2}(0 - 1) = -\frac{1}{2}$. This gives the desired expression (28). □

Theorem 3.1 states that the correlation function is constant on the disjoint sublattices L_n. The constant value only depends on the lattice-index n, and not on the dimension m or on (H, W). The lattices L_n themselves depend on m and H.

Figure 3 shows the different lattices L_i for the two-dimensional paperfolding sequences displayed in figure 2.

For similar paperfolding sequences p_1 and p_2, based on the $\mathbb{Z}$-similar binary number systems $(H_1, \{0, w_1\})$ and $(H_2, \{0, w_2\})$, respectively, the lattices L_i for p_1 and p_2, denoted by $\mathsf{L}_i^{p_1}$ and $\mathsf{L}_i^{p_2}$, respectively, relate by the consequence of theorem 2.2 as

$$\mathsf{L}_i^{p_2} = R\mathsf{L}_i^{p_1}.$$

As already mentioned, the spectrum is the Fourier transform of the correlation measure $\gamma_{pp} = \sum_{x \in \mathbb{Z}^m} \gamma(x)\delta_x$, regarded as a tempered distribution. In order to compute the Fourier transform, we state two technical lemmata.

Lemma 3.2 ([16] theorem 2.6)**.** *Let $B \in \mathbb{R}^{m \times m}$ be a matrix of rank m. The Fourier transform of the Dirac comb $L_B = \sum_{x \in B\mathbb{Z}^m} \delta_x$ is given by*

$$\widehat{L_B} = |\det(B)^{-1}| \sum_{\nu \in B^{-T}\mathbb{Z}^m} \delta_\nu,$$

where B^{-T} is the transposed inverse of B. The shifted Dirac comb $L_{B+\sigma} = \sum_{x \in B\mathbb{Z}^m + \sigma} \delta_x$, $\sigma \in \mathbb{R}^m$, has Fourier transform

$$\widehat{L_{B+\sigma}} = |\det(B)^{-1}| \sum_{\nu \in B^{-T}\mathbb{Z}^m} \exp(-2\pi \mathrm{i} \nu^T \sigma)\delta_\nu.$$

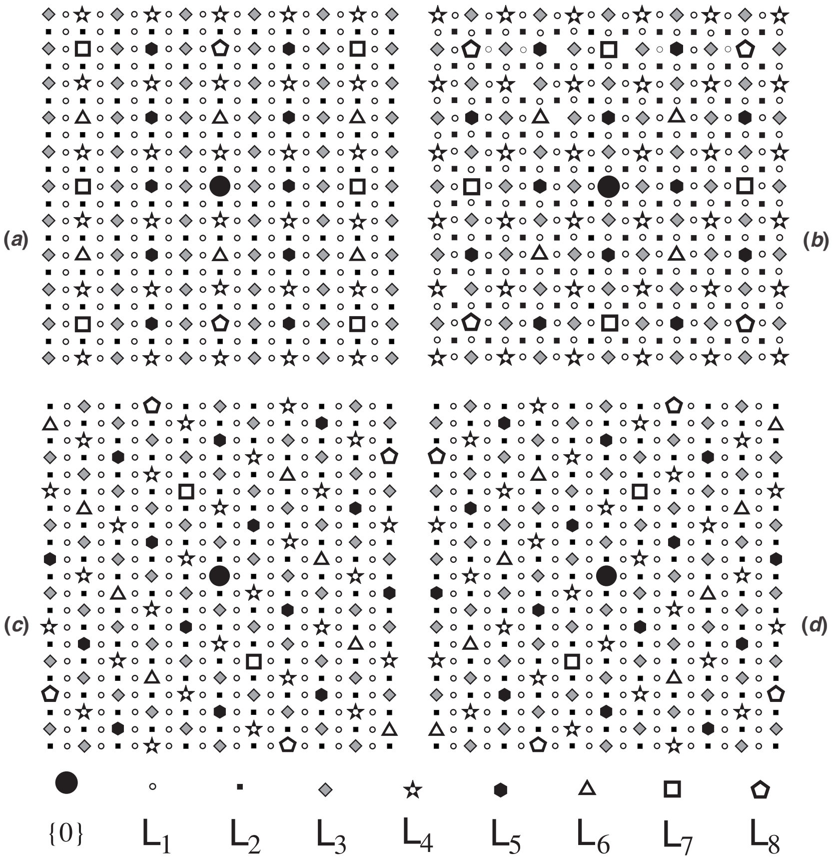


Figure 3. The lattices L_j, $j = 1, \ldots, 8$, restricted on the domain $[-10, 10]^2$. The correlation function of the two-dimensional paperfolding sequences is constant on each L_j. (*a*): For the companion binary number systems (C_1, E) and (C_3, E) (see (3)). The lattices L_j happen to coincide for these two cases, although the number systems are not $\mathbb{Z}$-similar. (*b*): For the binary number system $(H, E) \overset{\mathbb{Z}}{\sim} (C_1, E)$ (see (4)). The lattices in (*b*) can be obtained from (*a*) by a linear transformation. (*c*) and (*d*) For the number systems (C_2, E) and (C_4, E), respectively. Observe that both pictures are left–right (or up–down) reflections of each other.

Lemma 3.3. *Let $H \in \mathbb{Z}^{m\times m}$ be such that $|\det(H)| = 2$ and let $w, k \in \mathbb{Z}^m$, $w \notin H\mathbb{Z}^m$. Then the following is true: $k \notin H^T\mathbb{Z}^m$ if and only if*

$$\exp(-2\pi \mathrm{i} k^T H^{-1} w) = -1.$$

Proof. Let $\Psi(k) = \exp(-2\pi \mathrm{i} k^T H^{-1})w$. Since $|\det(H)| = 2$, $\Psi(k) = \exp(-\pi k^T \operatorname{Adj}(H))w$. Since $k^T \operatorname{Adj}(H)w$ is an integer for all $k \in \mathbb{Z}^m$, it follows that ± 1 are the only possible values for Ψ. If $k_2 = k_1 + H^T\xi$, where $k_1, k_2, \xi \in \mathbb{Z}^m$, then it follows that

$$\Psi(k_2) = \exp(-2\pi \mathrm{i}(k_1 + (H^T\xi))^T H^{-1} w) = \psi(k_1) \exp(-2\pi \mathrm{i}(H^T\xi)^T H^{-1} w = \Psi(k_1),$$

due to the fact that $(H^T\xi)^T H^{-1} w = \xi^T (H H^{-1}) w \in \mathbb{Z}$. As $\Psi(0) = 1$, it follows that, if $k \in H^T\mathbb{Z}^m$, then $\Psi(k) = 1$. Since Ψ is constant on residue classes mod H^T and since $|\det(H^T)| = 2$, it suffices to show that $\Psi(k) = -1$ for one $k \notin H^T\mathbb{Z}^m$.

Since $w \notin H\mathbb{Z}^m$, at least one component, say the jth one, of $H^{-1}w = \mathrm{Adj}(H)w/\det(H^T)$ is of the form $z + 1/2$ ($z \in \mathbb{Z}$). For e_j the jth unit vector in $\mathbb{Z}^m$ then one computes that $\Psi(e_j) = -1$. It follows that $e_j \notin H\mathbb{Z}^m$ and also that $\Psi(k) = -1$ for all $k \notin H^T\mathbb{Z}^m$. □

We are now prepared to compute the spectrum of the paperfolding sequences.

Theorem 3.4. *The spectrum of an* (H, W)*-automatic paperfolding sequence* p *with values* ± 1 *is given by*

$$\widehat{\gamma}_{pp} = \sum_{j=2}^{\infty} \frac{1}{4^{j-1}} \sum_{\nu \in \mathcal{L}_j} \delta_\nu,$$

where $\mathcal{L}_j = H^{-jT}(H^T\mathbb{Z}^m + w)$.

Proof. Due to (24), the correlation measure (as tempered distribution) $\gamma_{pp} = \sum_{x\in\mathbb{Z}^m} \gamma_{pp}(x)\delta_x$ can be written as

$$\gamma_{pp} = \sum_{x\in\mathbb{Z}^m} \gamma_{pp}(Hx)\delta_{Hx} + \sum_{x\in\mathbb{Z}^m} \gamma_{pp}(Hx + w)\delta_{Hx+w}.$$

Invoking relation (17), this reduces to

$$\gamma_{pp} = \sum_{x\in\mathbb{Z}^m} \gamma_{pp}(Hx)\delta_{Hx}.$$

By equation (16) this sum can be written as

$$\gamma_{pp} = \frac{1}{2} \sum_{x\in\mathbb{Z}^m} \gamma_{pp}(x)\delta_{Hx} + \frac{1}{2} \sum_{x\in\mathbb{Z}^m} \gamma_{gg}(x)\delta_{Hx}. \tag{29}$$

By (25), the first sum can be split as

$$\sum_{x\in\mathbb{Z}^m} \gamma_{pp}(Hx)\delta_{H^2x} + \sum_{x\in\mathbb{Z}^m} \gamma_{pp}(Hx + w)\delta_{H^2x+Hw}.$$

Again, due to (17), the second term equals 0, and thus, (29) becomes

$$\gamma_{pp} = \frac{1}{2} \sum_{x\in\mathbb{Z}^m} \gamma_{pp}(Hx)\delta_{H^2x} + \frac{1}{2} \sum_{x\in\mathbb{Z}^m} \gamma_{gg}(x)\delta_{Hx}.$$

Using the splitting of $\gamma_{pp}(Hx)$ as given in (16) again, one arrives at

$$\gamma_{pp} = \frac{1}{4} \sum_{x\in\mathbb{Z}^m} \gamma_{pp}(x)\delta_{H^2x} + \frac{1}{4} \sum_{x\in\mathbb{Z}^m} \gamma_{gg}(x)\delta_{H^2x} + \frac{1}{2} \sum_{x\in\mathbb{Z}^m} \gamma_{gg}(x)\delta_{Hx}.$$

An N-time application of this splitting gives

$$\gamma_{pp} = \frac{1}{2^N} \sum_{x\in\mathbb{Z}^m} \gamma_{pp}(x)\delta_{H^Nx} + \sum_{j=1}^{N} \left(\frac{1}{2^j} \sum_{x\in\mathbb{Z}^m} \gamma_{gg}(x)\delta_{H^jx} \right).$$

Note that

$$\lim_{\mathbb{N}\to\infty} \frac{1}{2^N} \sum_{x\in\mathbb{Z}^m} \gamma_{pp}(x)\delta_{H^Nx} = 0$$

in distribution sense. Hence,

$$\gamma_{pp} = \lim_{N\to\infty} \sum_{j=1}^{N} \left(\frac{1}{2^j} \sum_{x\in\mathbb{Z}^m} \gamma_{gg}(x)\delta_{H^jx} \right).$$

Thus, the spectrum $\widehat{\gamma}_{pp}$ is given by the Fourier transform

$$\widehat{\gamma}_{pp} = \lim_{N\to\infty} \sum_{j=1}^{N} \widehat{\left(\frac{1}{2^j} \sum_{x\in\mathbb{Z}^m} \gamma_{gg}(x)\delta_{H^j x} \right)}$$

Setting

$$Q_N = \sum_{j=1}^{N} \left(\frac{1}{2^j} \sum_{x\in\mathbb{Z}^m} \gamma_{gg}(x)\delta_{H^j x} \right),$$

it holds that

$$\widehat{\gamma}_{pp} = \lim_{N\to\infty} \widehat{Q}_N, \tag{30}$$

in the distribution sense. In order to calculate $\widehat{Q}_N$, we use (24) to obtain

$$\sum_{x\in\mathbb{Z}^m} \gamma_{gg}(x)\delta_{H^j x} = \sum_{x\in\mathbb{Z}^m} \gamma_{gg}(Hx)\delta_{H^{j+1}x} + \sum_{x\in\mathbb{Z}^m} \gamma_{gg}(Hx+w)\delta_{H^j(Hx+w)}.$$

Using equations (18) and (19) this becomes

$$\sum_{x\in\mathbb{Z}^m} \gamma_{gg}(x)\delta_{H^j x} = \sum_{x\in\mathbb{Z}^m} (\delta_{H^{j+1}x} - \delta_{H^j(Hx+w)}).$$

The Fourier transform of this Dirac comb can be obtained by applying lemma 3.2 with $B = H^{j+1}$ and $\sigma = H^j w$. This gives the Fourier transform

$$\widehat{Q}_N = \sum_{j=1}^{N} \frac{1}{2^{2j+1}} \sum_{\nu\in H^{-(j+1)T}\mathbb{Z}^m} (1 - \exp(-2\pi \mathrm{i}\nu^T H^j w))\delta_\nu.$$

Thus, (30) becomes

$$\widehat{\gamma}_{pp} = \sum_{j=1}^{\infty} \frac{1}{2^{2j+1}} \sum_{\nu\in H^{-(j+1)T}\mathbb{Z}^m} (1 - \exp(-2\pi \mathrm{i}\nu^T H^j w))\delta_\nu. \tag{31}$$

In other words, $\widehat{\gamma}_{pp}$ is discrete with (possible) peaks at $H^{-jT}\mathbb{Z}^m$, $j \geqslant 2$. We will now calculate the weight of the Dirac peaks at these positions.

For $\nu \in \mathbb{Z}^m$, the above formula shows that $\widehat{\gamma}_\nu(0) = 0$, i.e., the spectrum has no peaks on the lattice $\mathbb{Z}^m$.

Now let $\nu = H^{-j_0T}(H^T\eta + w) \in H^{-j_0T}\mathbb{Z}^m$ for an $\eta \in \mathbb{Z}^m$ and a $j_0 \geqslant 2$, then there are two possibilities:

(a) $\nu \notin H^{-jT}\mathbb{Z}^m$ for $j < j_0$, if not then $H^{-j_0T}(H^T\eta + w) \in H^{-jT}\mathbb{Z}^m$, or $H^T\eta + w \in H^{(j_0-j)T}\mathbb{Z}^m$ with $(j_0 - j) \geqslant 0$, which contradicts the fact that $w \notin H\mathbb{Z}^m$.

(b) $\nu \in H^{-jT}\mathbb{Z}^m$ for all $j \geqslant j_0$, as $H^{-kT}\mathbb{Z}^m \subset H^{-lT}\mathbb{Z}^m$ for $k < l$.

Then it follows from (31) that the sum of the peak intensities at this ν is given by

$$\sum_{l=j_0-1}^{\infty} \frac{1}{2^{2(j_0+l-1)+1}} (1 - \exp(-2\pi \mathrm{i}(H^{-j_0T}(H^T\eta + w))^T H^{j_0+l-1}w))$$

$$= \sum_{l=0}^{\infty} \frac{1}{2^{2(l+j_0-1)+1}} (1 - \exp(-2\pi \mathrm{i}(H^{-j_0T}(H^T\eta + w))^T H^{l+j_0-1}w)). \tag{32}$$

The exponent in this expression can be written as $-2\pi\mathrm{i}(\eta^T H + w^T)H^l H^{-1}w$. By lemma 3.3, it follows that

$$1 - \exp(-2\pi \mathrm{i}(\eta^T H + w^T)H^l H^{-1}w) = \begin{cases} 2 & \text{if} \quad l = 0 \\ 0 & \text{otherwise.} \end{cases}$$

Using this in (32) shows that the weight of the Dirac peak at $\nu = H^{-j_0 T}(H^T\eta + w)$ equals $1/2^{2(j_0-1)}$, independent of η. Note further that the set $\{x|x = H^{-j_0 T}(H^T\eta + w), \eta \in \mathbb{Z}^m\}$, $j_0 \geqslant 2$ is precisely the point lattice $\mathcal{L}_{j_0}$, and therefore

$$\widehat{\gamma}_{pp} = \sum_{j=2}^{\infty} \frac{1}{4^{j-1}} \sum_{\nu\in\mathcal{L}_j} \delta_\nu. \qquad \square$$

Theorem 3.4 reflects the result in [1], p 345, for the classical ±1-paperfolding sequence over $\mathbb{N}$, and gives a generalization to the class of automaton-similar ±1-valued paperfolding sequences.

If g is automaton-similar to a ±1-paperfolding sequence, then one has

Corollary 3.5. *Let g be automaton-similar to a paperfolding sequence p with values ±1, i.e., g is given by*

$$g(x) = \begin{cases} a & \text{if} \quad p(x) = -1 \\ b & \text{if} \quad p(x) = +1 \end{cases} \qquad \text{with} \quad a, b \in \mathbb{C}, a \neq b.$$

The correlation function of g is

$$\gamma_{gg}(k) = \left(\frac{b-a}{2}\right)^2 \gamma_{pp}(k) + \left(\frac{a+b}{2}\right)^2$$

and the spectrum of g is

$$\widehat{\gamma}_{gg} = \left(\frac{b-a}{2}\right)^2 \widehat{\gamma}_{pp} + \left(\frac{a+b}{2}\right)^2 \sum_{x\in\mathbb{Z}^m} \delta_x.$$

Proof. Define $\theta : \mathbb{C} \to \mathbb{C}$ as $\theta(z) = \left(\frac{b-a}{2}\right)z + \left(\frac{a+b}{2}\right)$. Then $\theta(-1) = a$ and $\theta(1) = b$, and $g(x) = \left(\frac{b-a}{2}\right)f(x) + \left(\frac{a+b}{2}\right)$. The assertion now follows from lemma 2.4, with $\mu_p = 0$, see (22). $\square$

Thus, the spectrum of a sequence which is automaton-similar but not similar to a ±1-paperfolding sequence has additional peaks on the lattice $\mathbb{Z}^m$. A particular example is obtained by replacing −1 by 0 and keeping the +1. Then the resulting sequence p' is the characteristic sequence of the Delone set $\mathsf{P} = \{x|p'(x) = 1\} \subset \mathbb{Z}^2$, see [8] for details. P will be called the *paperfolding set*. The correlation function and diffraction spectrum of this set are then given by

$$\gamma_{p'p'}(k) = \frac{1}{4}\gamma_{pp}(k) + \frac{1}{4} \qquad \text{for} \quad k \in \mathbb{Z}^m$$

$$\widehat{\gamma}_{p'p'} = \frac{1}{4}\widehat{\gamma}_{pp} + \frac{1}{4}\sum_{\nu\in\mathbb{Z}^m} \delta_\nu.$$

Also note that here the lattices $\mathcal{L}_i^{p_1}$ and $\mathcal{L}_i^{p_2}$ for two similar paperfolding sequences p_1 and p_2 relate by application of theorem 2.3 as

$$\mathcal{L}_i^{p_2} = R^{-T}\mathcal{L}_i^{p_1}.$$

Note also that on the lattice $\mathcal{L}_1 = \mathbb{Z}^m + H^{-T}w$ no diffraction peaks appear for any of the paperfolding sequences.

Figure 4 gives an idea about the diffraction spectrum of the paperfolding sequences displayed in figure 2.

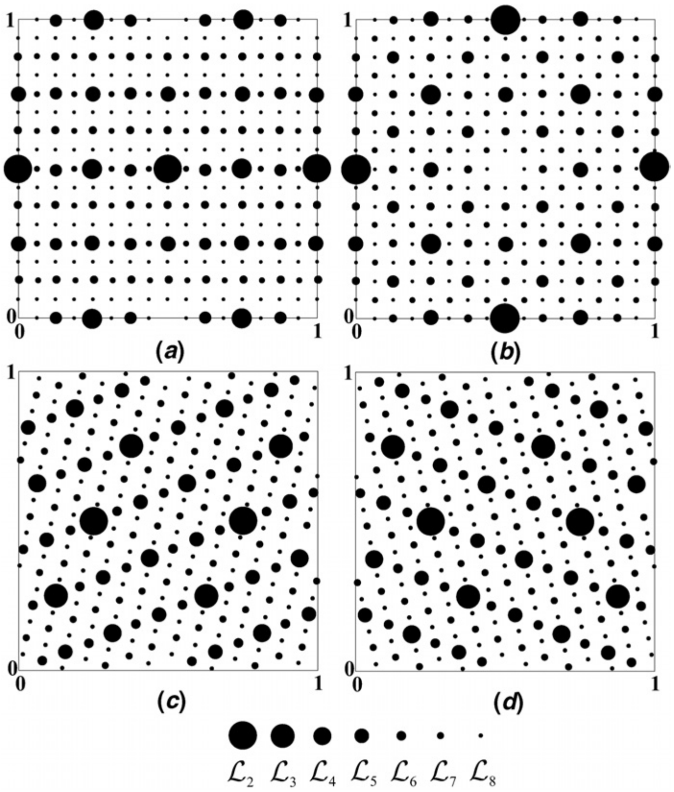


Figure 4. The lattices $\mathcal{L}_j$, $j = 2, \ldots 8$, restricted to the interval $[0, 1]^2$, on which the diffraction spectrum of the ± 1-paperfolding sequences, displayed in figure 2, is constant. The size of the circles increases with the weight of the peaks. (*a*) For the binary number systems (C_1, E) and (C_3, E). (*b*) For the binary number system $(H, E) \overset{\mathbb{Z}}{\sim} (C_1, E)$ (see(4)). The lattices in (*b*) can be obtained from (*a*) by a linear transformation. (*c*) and (*d*) For the binary number systems (C_2, E) and (C_4, E), respectively. Observe that both pictures are left–right (or up–down) reflections of each other (just as for the correlation lattices).

4. Correlation and spectral properties of Rudin–Shapiro sequences

In this section,we study the automaton-similarity class of the Rudin–Shapiro sequence. Like in the case of the paperfolding sequence,we begin with m-dimensional ± 1-valued Rudin–Shapiro sequences. We shall show that the spectral measure of such a sequence is always the Lebesgue measure on $\mathbb{R}^m$.

The decimation matrices and the kernel graph for an (H, W)-automatic Rudin–Shapiro sequence r with values ± 1 are displayed in figure 1. One can easily see that r is also given by the recursive relations

$$r(Hx) = r(x), \qquad r(H^2x + w) = r(x), \qquad r(H^2x + Hw + w) = -r(Hx + w),$$

for $x \in \mathbb{Z}^m$ and with $r(0) = -1$. Note that these relations also hold for the automaton-similar sequence $-r$. Using this, one sees that the four kernel elements r, s, t, u are related by $t = -s$, $u = -r$. The corresponding correlation vector Γ_r in (12) and (13) is the 16-component vector

$$(\gamma_{rr}, \gamma_{rs}, \gamma_{rt}, \gamma_{ru}, \gamma_{sr}, \gamma_{ss}, \gamma_{st}, \gamma_{su}, \gamma_{tr}, \gamma_{ts}, \gamma_{tt}, \gamma_{tu}, \gamma_{ur}, \gamma_{us}, \gamma_{ut}, \gamma_{uu})^T.$$

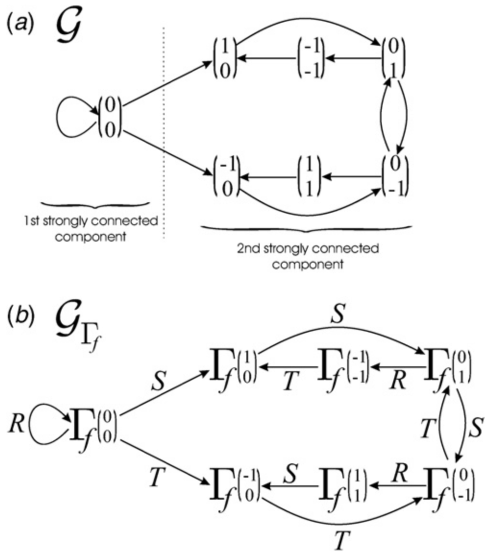


Figure 5. The graphs $\mathcal{G}$ and $\mathcal{G}_{\Gamma_f}$ for an (H, W)-automatic sequence f with $H = \begin{pmatrix} -1 & -1 \\ 1 & -1 \end{pmatrix}$ and $W = \{(0,0)^T, (1,0)^T\}$.

It was already derived in [12] that $\Gamma_r(0)$ exists and equals

$$\Gamma_r(0) = (1, 0, 0, -1, 0, 1, -1, 0, 0, -1, 1, 0, -1, 0, 0, 1)^T, \tag{33}$$

and that, as a consequence, $\Gamma_r(k)$ exists for all $k \in \mathbb{Z}^m$ and can be calculated using equations (12) and (13).

Theorem 4.1. *Any m-dimensional* ± 1*-valued Rudin–Shapiro sequence has the correlation function*

$$\gamma_{rr}(0) = 1$$
$$\gamma_{rr}(k) = 0 \qquad \text{for} \quad k \neq 0.$$

Correspondingly, the diffraction spectrum is absolutely continuous, i.e., for $\nu \in \mathbb{R}^m$,

$$\widehat{\gamma}_{rr}(\nu) = 1.$$

Proof. Rewrite equations (12) and (13) in the form

$$\Gamma_f(k) = \tfrac{1}{2}(A_0 \otimes A_0 + A_w \otimes A_w)\Gamma_f(H^{-1}k) \qquad \text{if} \quad k \in H\mathbb{Z}^m$$
$$\Gamma_f(k) = \tfrac{1}{2}(A_0 \otimes A_w)\Gamma_f(H^{-1}(k-w)) + \tfrac{1}{2}(A_w \otimes A_0)\Gamma_f(H^{-1}(k+w)) \qquad \text{if} \quad k \in H\mathbb{Z}^m + w.$$

Or, for further convenience, we write this as

$$\begin{aligned} \Gamma_f(k) &= R\Gamma_f(H^{-1}k) && \text{if} \quad k \in H\mathbb{Z}^m, \\ \Gamma_f(k) &= S\Gamma_f(H^{-1}(k-w)) + T\Gamma_f(H^{-1}(k+w)) && \text{if} \quad k \in H\mathbb{Z}^m + w. \end{aligned} \tag{34}$$

This shows that $\Gamma_f(k)$ depends either on $\Gamma_f(H^{-1}k)$ or on $\Gamma_f(H^{-1}(k-w))$ and $\Gamma_f(H^{-1}(k+w))$. One can display these dependences in a directed graph $\mathcal{G}$ with vertices $k \in \mathbb{Z}^m$ that have one incoming edge from $H^{-1}k$ if $k \in H\mathbb{Z}^m$, and two incoming edges, one from $H^{-1}(k-w)$ and the other from $H^{-1}(k+w)$ if $k \in H\mathbb{Z}^m + w$.

An illustration is given in figure 5(a) for the binary number system (H, E) given in (4). The graph in the illustration has two strongly connected components (recall that a strongly

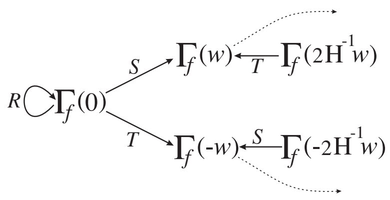


Figure 6. The universal part of the dependency graph $\mathcal{G}_\Gamma$ for automatic sequences based on binary number systems.

connected component is a subgraph such that for any two vertices a, b there exists a directed path from a to b in this subgraph).

Note that {0} always forms a strongly connected component. In the case at hand there is a second strongly connected component. The vertices Ξ of the strongly connected components are given by the points

$$\xi_1 = (1,0)^T \qquad \xi_2 = (0,1)^T \qquad \xi_3 = (-1,0)^T \qquad \xi_4 = (0,-1)^T$$
$$\xi_5 = (1,1)^T \qquad \xi_6 = (-1,-1)^T \qquad \xi_7 = (0,0)^T.$$

The graph $\mathcal{G}$ induces a second graph $\mathcal{G}_{\Gamma_f}$ with $\Gamma_f(k)$ as vertices, and directed edges labelled R, S, T which express the proper multiplications in (34), see figure 5(*b*).

As shown in [12], the existence of a finite number of finite strongly connected components is a universal feature for any number system (H, W). For an (H, W)-automatic sequence f it was shown that, if Ξ denotes the vertices of all connected components, then all $\Gamma_f(k)$ for $k \notin \Xi$ can be calculated from the values at $k \in \Xi$ in a recursive way. Let $\Xi^* = \Xi\backslash\{0\}$ and consider $(\Gamma_f(k))_{\xi\in\Xi^*}$ as a $\Xi^* \times (\ker(f) \times \ker(f))$-vector. Then $(\Gamma_f(k))_{k\in\Xi^*}$ is the unique fixed point of the contractive map (cf [12] equation (29))

$$x \mapsto B_\Xi x + C_\Xi \Gamma_f(0), \tag{35}$$

where B_Ξ is the $\Xi^* \times (\ker(f) \times \ker(f))$-block matrix reflecting the corresponding transfers between $\Gamma_f(\xi)$ and $\Gamma_f(H^{-1}\xi)$ or $\Gamma_f(H^{-1}(\xi - w))$ and $\Gamma_f(H^{-1}(\xi + w))$ as displayed in the graph $\mathcal{G}_{\Gamma_f}$.

$C_\Xi\Gamma_f(0)$ corresponds to the connections between $\Gamma_f(0)$ and all other $\Gamma_f(\xi)$, $\xi \in \Xi^*$. For binary number systems (H, W), this connection is universal and is depicted in figure 6. It corresponds to the equations

$$\Gamma_f(0) = R\Gamma_f(0) \tag{36}$$

$$\Gamma_f(w) = S\Gamma_f(0) + T\Gamma_f(2H^{-1}w) \tag{37}$$

$$\Gamma_f(-w) = S\Gamma_f(-2H^{-1}w) + T\Gamma_f(0). \tag{38}$$

There are no other links between $\Gamma_f(0)$ and the other $\Gamma_f(k)$, $k \in \mathbb{Z}^m\backslash\{0\}$. The last two of these equations correspond to $C_\Xi\Gamma_f(0)$ in (35).

We apply this to the Rudin–Shapiro sequence r. The crucial point is that for any ± 1 sequence in the Rudin–Shapiro class the following equations hold:

$$S\Gamma_r(0) = 0 \qquad \text{and} \qquad T\Gamma_r(0) = 0, \tag{39}$$

with S and T as given in (33). Since the Rudin–Shapiro sequence satisfies (39) it follows that in equations (37) and (38) the values of $\Gamma_f(\pm w)$ are independent of $\Gamma_r(0)$. This implies that $C_\Xi\Gamma_r(0) = 0$. Then (35) becomes the contracting map

$$x \mapsto B_\Xi x,$$

and $(\Gamma_r(\xi))_{\xi\in\Xi^*}$ equals the unique fixed point $x^* = 0$ of this map. Thus $\Gamma_r(\xi) = 0$ for $\xi \in \Xi^*$, and as all other $\Gamma_r(k)$ with $k \notin \Xi^*$ are determined by $\Gamma_r(\xi), \xi \in \Xi^*$ it follows that $\Gamma_r(k) = 0$, and thus also that $\gamma_{rr}(k) = 0$, for all $k \in \mathbb{Z}^m\backslash\{0\}$. As we already know that $\gamma_{rr}(0) = 1$, this proves the correlation part of the theorem. The correlation measure is thus a Dirac impulse at 0, and hence the spectrum has constant value 1. □

Remark. It also follows from the proof of the above theorem that all kernel elements of r, namely the sequences $s, t = -s, u = -r$, have the same autocorrelation function and spectrum as r. As for the crosscorrelation functions, it is clear that $\gamma_{rs}(k) = \gamma_{sr}(k) = \gamma_{rt}(k) = \gamma_{tr}(k) = \gamma_{su}(k) = \gamma_{us}(k) = \gamma_{ut}(k) = \gamma_{tu}(k) = 0$ for all $k \in \mathbb{Z}^m$ and that $\gamma_{ru}(k) = \gamma_{ur}(k) = \gamma_{st}(k) = \gamma_{ts}(k) = -1$ if $k = 0$, 0 if $k \neq 0$.

The result of theorem 4.1 also generalizes a similar result for the one-dimensional Rudin–Shapiro sequence defined on $\mathbb{N}$, see [18] giving credit to Kamae.

Corollary 4.2. *Let g be automaton-similar to a ± 1-valued Rudin–Shapiro sequence r, i.e., g is given by*

$$g(x) = \begin{cases} a & \text{if} \quad r(x) = -1 \\ b & \text{if} \quad r(x) = +1 \end{cases} \qquad \text{with} \quad a, b \in \mathbb{C}, a \neq b.$$

The correlation function of g is

$$\gamma_{gg}(0) = \left(\frac{b-a}{2}\right)^2 \gamma_{rr}(0) + \left(\frac{a+b}{2}\right)^2 \qquad \text{for} \quad k = 0$$

$$\gamma_{gg}(k) = \left(\frac{a+b}{2}\right)^2 \qquad \text{for} \quad k \in \mathbb{Z}^m\backslash\{0\}.$$

and the spectrum of g is

$$\widehat{\gamma}_{gg} = \left(\frac{b-a}{2}\right)^2 + \left(\frac{a+b}{2}\right)^2 \sum_{x\in\mathbb{Z}^m} \delta_x.$$

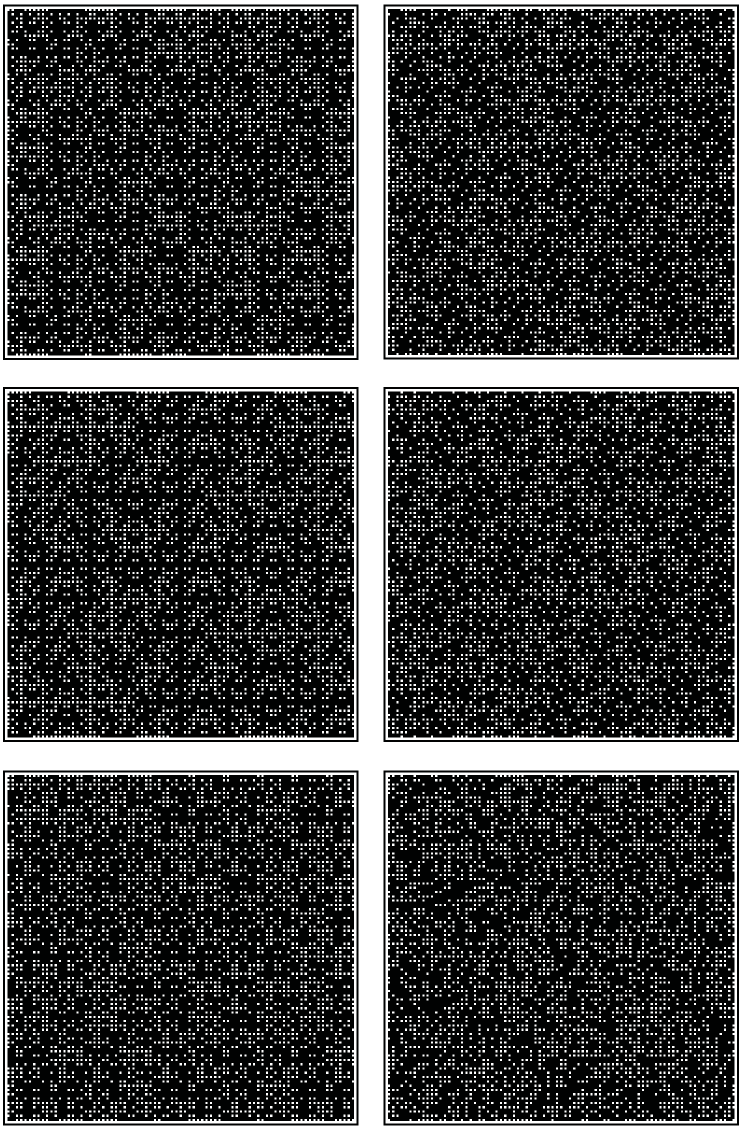
The proof is similar to the proof of corollary 3.5 and uses the nontrivial result that the average μ_r of any ± 1-valued sequence in the automaton-similarity class of Rudin–Shapiro sequences satisfies $\mu_r = 0$, see [17].

If r' is the Rudin–Shapiro sequence obtained from r by replacing -1 by 0 while keeping the +1-value, then r' can be considered as the characteristic sequence of the Delone set $\mathsf{R} = \{x \in \mathbb{Z}^m | r'(x) = 1\}$ which can be named Rudin–Shapiro set. Corollary 4.2 gives

$$\gamma_{r'r'}(0) = \frac{1}{2}, \qquad \gamma_{r'r'}(k) = \frac{1}{4} \qquad \text{for} \quad k \in \mathbb{Z}^m\backslash\{0\}$$

$$\widehat{\gamma}_{r'r'} = \frac{1}{4} + \frac{1}{4}\sum_{\nu\in\mathbb{Z}^m} \delta_\nu.$$

By the fact that the ± 1-valued Rudin–Shapiro sequences have the same correlation function and spectrum as a pure i.i.d. random sequence with values ± 1, it has often been said, at least for the one-dimensional case, that this deterministic Rudin–Shapiro sequence mimicks randomness (is a pseudo-random sequence), see [1] and the references therein. Although a plain graphical representation of a one-dimensional Rudin–Shapiro sequence gives no particular visual indication about its nonrandomness, a glimpse at the graphical representation of higher dimensional Rudin–Shapiro sequence hints at the presence of a structure which is



(*a*) (*b*) (*c*) (*d*) (*e*) (*f*)

Figure 7. (*a*)–(*d*) The two-dimensional Rudin–Shapiro sequences based on the four canonical $\mathbb{Z}^2$-binary number systems (C_j, E), $j = 1, 2, 3, 4$, in that order (see (3)). Similar representation as in figure 2. Part (e) displays the Rudin–Shapiro set for the number system $(H, E) \overset{\mathbb{Z}}{\sim} (C_1, E)$ (see (4)). It means that (*e*) can be obtained from (*a*) by a $\mathbb{Z}$-linear coordinate transformation (cf theorem 2.1, with $R = Q_H$ as given in (4)). (*f*) A completely random pattern.

certainly nonrandom. This is demonstrated in figure 7 displaying the two-dimensional $\{0, 1\}$-valued Rudin–Shapiro sequences (sets) for the companion binary number systems (C_j, E) and (H, E) in $\mathbb{Z}^2$, together with a random pattern. The Rudin–Shapiro sequences have some 'quasi-periodicity' or long-range order along certain directions. This becomes even clearer when we compare the sequence with a shifted version of itself in a so-called (h, v)-*shift-coincidence pattern*. For a $(0, 1)$-valued two-dimensional sequence r, and $h, v \in \mathbb{Z}$, it is defined as $r_{h,v}(x, y) = |r(x, y) - r(x + h, y + v)|$ which is 0 if $r(x, y)$ and $r(x + h, y + v)$

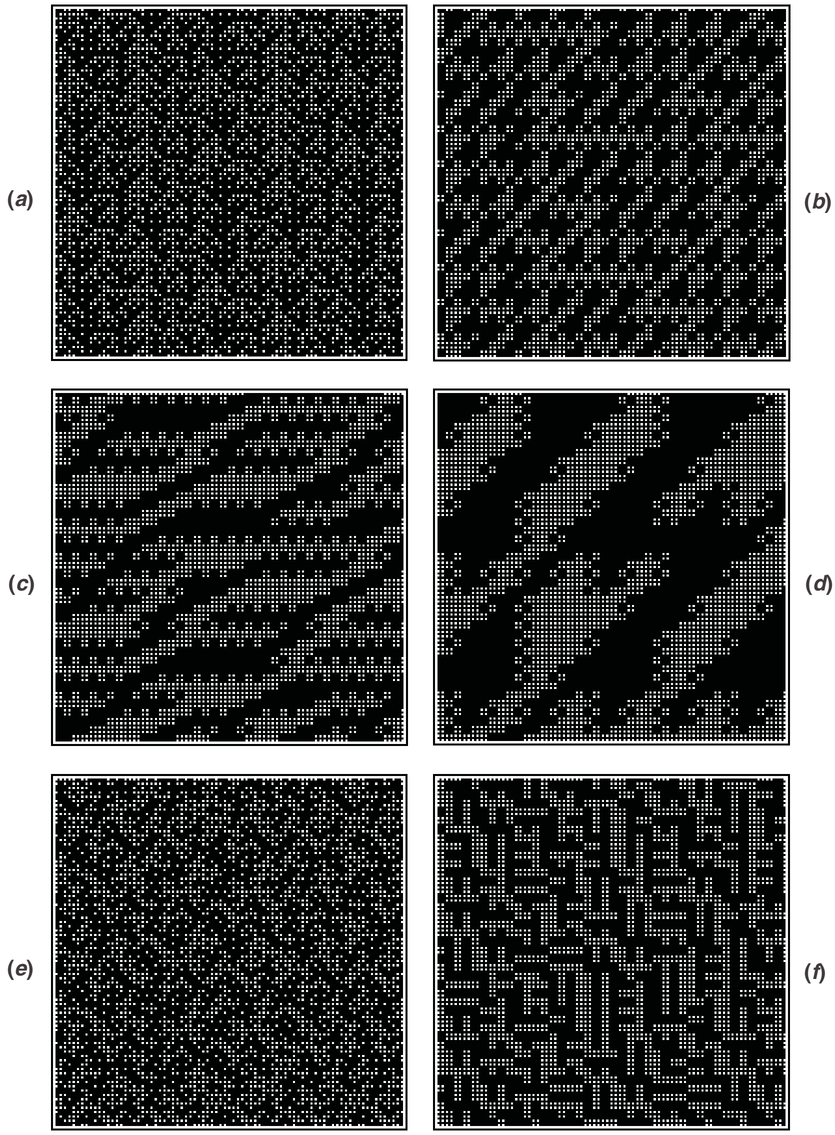


Figure 8. Some shift-coincidence patterns $r_{h,v}$ for the Rudin–Shapiro sequence r in figure 7(a). Graphical representation as illustrated in figure 2(f). Black areas indicate coincidences between r and its (h, v)-shifted version, white areas anti-coincidences. (a) $(h, v) = (3, 0)$; (b) $(h, v) = (4, 0)$; (c) $(h, v) = (8, 0)$; (d)$(h, v) = (16, 0)$; (e)$(h, v) = (3, 7)$; (f) $(h, v) = (52, 2)$.

have the same value (or have coinciding values), and otherwise 1. Figure 8 shows a few shift-coincidence patterns for the Rudin–Shapiro sequence in figure 7(a). All these patterns, no matter what the shift is, uncover a long-range correlation in the underlying Rudin–Shapiro sequence. For $(h, 0)$-, $(0, v)$- and (h, v)-shifts with $h = v = 2^k$, the black coincidence areas and the white anti-coincidence areas increase in size when k increases (compare (b), (c), (d)). The fact that the correlation function satisfies $\gamma_{rr}(k) = 0$ for $k \in \mathbb{Z}^2\backslash\{0\}$ then means that the number of coincidences equals the number of anti-coincidences, no matter what the shifts

(h, v) are. This also holds for the higher-dimensional cases, where a similar shift-coincidence sequence can be considered.

Conclusion, open questions. We have shown that the elements in the automaton-similar class of the ± 1-paperfolding sequence have essentially the same discrete spectrum. For the sequences in the automaton-similar class of the ± 1-Rudin–Shapiro sequence, the spectral measure is essentially the Lebesgue measure. This could suggest that the spectral properties of automaton-similar sequences only depend on the structure of the underlying automaton. However, this is not true. As preliminary work on the automaton-similarity class of the Thue–Morse sequence indicates, there are two possible spectra for this case. Depending on the binary number system, a Thue–Morse sequence may either be periodic, inducing a discrete spectrum, or it may have a singular continuous spectrum like the ± 1-Thue–Morse sequence over $\mathbb{N}$. Details are subject of a forthcoming paper.

A closer inspection of the proof of theorem 3.4 reveals, letting aside a few technicalities, that any automatic sequence whose kernel graph has a sink that corresponds to kernel graphs of periodic sequences has a purely discrete spectrum.

On the other hand, the situation for the Rudin–Shapiro class seems to be extremely singular. The reason for the Lebesgue measure to be the spectrum lies in the validity of the 'uncoupling' condition (39). So far no automatic sequence (based on binary number systems), besides those of the Rudin–Shapiro class studied above, has been found for which the 'uncoupling' condition (39) holds. It remains an open question whether there are any. Further investigations are needed in order to find out whether candidates can be found among possible higher-dimensional versions of the one-dimensional generalized Rudin–Shapiro sequences presented in [19, 20].

Acknowledgments

This research was supported by the Concerted Action Project GOA-Mefisto of the Flemisch Community, by the FWO (Fund for Scientific Research Flanders) project G.0121.03 and by the Belgian Program on Interuniversity Attraction Poles of the Belgian Prime Minister's Office for Science, Technology and Culture (IAP V-22).

References

[1] Allouche J P and Mendès France M 1995 Automata and automatic sequences *Beyond Quasicristals* ed F Axel and D Gratias (Heidelberg: Springer Les Éditions de Physique Les Ulis) p 293

[2] Queffélec M 1987 *Substitution Dynamical Systems—Spectral Analysis (Lecture Notes in Mathematics* vol 1294*)* (Berlin: Springer)

[3] Höffe M and Baake M 2000 Surprises in diffuse scattering *Z. Kristallogr.* **215** 441

[4] Frank Priebe N 2003 Substitution seqencesin $\mathbb{Z}^d$ with non-simple Lebesgue component in the spectrum *Ergod. Th. Dyn. Syst.* **23** 519

[5] Frank Priebe N 2004 Multidimensional constant-length substitution sequences *Topology and its Applications* (special issue) *Proc. of the Conf. in Dynamical Systems at UNT* (at press)

[6] Baake M and Moody R V (ed) 2000 *Directions in Mathematical Quasicrystals* (Providence, RI: American Mathematical Society)

[7] Senechal M 1995 *Quasicrystals and Geometry* (Cambridge: Cambridge University Press)

[8] Barbé A and von Haeseler F 2004 Automatic sets and Delone sets *J. Phys. A: Math. Gen.* **37** 4017

[9] Allouche J P and Shallit J 2003 *Automatic Sequences. Theory, Applications, Generalizations* (Cambridge: Cambridge University Press)

[10] von Haeseler F 2003 *Automatic Sequences* (Berlin: de Gruyter)

[11] Barbé A and von Haeseler F 2004 Binary number systems for $\mathbb{Z}^k$ SCD/SISTA Report No. 04-90 (submitted)

[12] Barbé A and von Haeseler F 2004 Correlation functions of higher-dimensional automatic sequences *J. Phys. A: Math. Gen.* **37** 10879
[13] Gasquet C and Witomski P 1999 *Fourier Analysis and Applications* (New York: Springer)
[14] Baake M 2002 Diffraction of weighted lattice subsets *Can. Math. Bull.* **45** 483
[15] Donoghue W F Jr 1969 *Distributions and Fourier Transforms* (New York: Academic)
[16] Lagarias J C 2000 Mathematical quasicrystals and the problem of diffraction *Directions in Mathematical Quasicrystals* ed M Baake and R V Moody (Providence, RI: American Mathematical Society)
[17] Barbé A and von Haeseler F 2004 Averages of automatic sequences SCD/SISTA Report No. 04-223 (submitted)
[18] Queffélec M 1987 Une nouvelle propriété des suites de Rudin–Shapiro *Ann. Inst. Fourier (Grenoble)* **37** 115
[19] Allouche J P and Liardet P 1991 Generalized Rudin–Shapiro sequences *Acta Arith.* **60** 1
[20] Mendés France M and Tenenbaum G 1981 Dimension des courbes planes, papiers pliés et suites de Rudin–Shapiro *Bull. Soc. Math. Fr.* **109** 207